# A Modular Completeness Proof for the Superposition Calculus

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**Abstract.** This paper describes a detailed completeness proof for Bachmair's and Ganzinger's superposition calculus. The proof follows the modularization steps given by Waldmann et al.'s saturation framework, and serves as a blueprint for a formalized completeness proof of superposition in the interactive proof assistant Isabelle/HOL on top of the existing formalization of the saturation framework in Isabelle/HOL.

## 1 Background

Bachmair's and Ganzinger's superposition calculus [2, 3] is the standard calculus for automated theorem proving in quantified equational first-order logic. It forms the basis of state-of-the-art theorem provers like Vampire [5] and E [7].

Superposition is one example of a saturation calculus – a calculus that gets a set of formulas (typically clauses) as inputs and processes this set by (i) deriving new formulas from the given ones and adding them to the formula set and by (ii) deleting formulas that turn out to be superfluous. This process is repeated until either a contradiction is found (which then demonstrates that the original input set was contradictory) or until a saturated set of formulas is reached to which no further formulas need to be added anymore. The calculus is called refutationally complete if it has the property that every fair derivation starting from a contradictory input set eventually detects the contradiction.

Waldmann et al. [9] described a framework to prove the refutational completeness of saturation calculi. The framework was formalized (and extended) in the interactive proof assistent Isabelle/HOL [6] by Tourret [8] and Blanchette and Tourret [4]. In the current paper, we present a detailed refutational completeness proof for the superposition calculus that follows the modularization steps given in [9] and that can therefore serve as a blueprint for a formalized completeness proof for the superposition calculus in Isabelle/HOL on top of the work described in [4, 8].

## 2 Preliminaries

We assume that the reader is familiar with basic concepts in first-order logic and with the saturation framework [9]. For standard results and notations in term rewriting, we refer to Baader and Nipkow [1].

A first-order *signature* is a triple $\Sigma = (\Xi, \Omega, \Pi)$, where $\Xi$ is a set of *sorts*, $\Omega$ is set of *function symbols*, and $\Pi$ is set of *predicate symbols*. (All sets are disjoint.) Every function symbol $f \in \Omega$ and predicate symbol $P \in \Pi$ has a unique declaration $f : \xi_1 \dots \xi_n \to \xi_0$ and $P : \xi_1 \dots \xi_n$, $n \geq 0$, $\xi_j \in \Xi$. Furthermore let $X$ be a $\Xi$-sorted set of *variables* (disjoint from $\Xi$, $\Omega$, $\Pi$). Then $\mathcal{T}_\Sigma(X)$ is the set of (well-sorted) *terms* over $\Sigma$ and $X$.

A (well-sorted) *equation* is an unordered pair $(s, t)$ of terms with the same sort, usually written as $s \approx t$. A (well-sorted) *non-equational atom* over $\Sigma$ and $X$ has the form $P(t_1, \dots, t_n)$ with $P \in \Pi$ and $t_i \in \mathcal{T}_\Sigma(X)$. An *atom* is an equation or a non-equational atom. A *literal* is an atom $A$ or a negated atom $\neg A$. We usually write $s \not\approx t$ instead of $\neg(s \approx t)$. A *clause* is a multiset of literals, usually written as a disjunction. The symbol $\bot$ denotes the empty clause, that is, false.

In the sequel, all terms, atoms, literals, equations, substitutions are assumed to be well-sorted.

To simplify the presentation, we will assume from now on that $\Pi = \emptyset$ and that predicate symbols $P : \xi_1 \dots \xi_n$ are replaced by function symbols $f_P : \xi_1 \dots \xi_n \to \mathsf{bool}$, so that non-equational literals $[\neg]\, P(t_1, \dots, t_n)$ are encoded as equations $[\neg]\, f_P(t_1, \dots, t_n) \approx \mathsf{true}$.

We assume that $\succ$ is a reduction ordering on terms that is total on ground terms and has the subterm property on ground terms i.e., $t[s]_p \succ s$ if $p \neq \varepsilon$. (In the single-sorted case, the subterm property follows from totality on ground terms, compatibility with contexts, and well-foundedness, but in the multi-sorted case, we have to require it explicitly.)

## 3 (Ground) Inference System

Let a fixed first-order signature $\Sigma = (\Xi, \Omega, \Pi)$ be given. We define $\mathbf{G}$ as the set of ground first-order clauses over $\Sigma$ and $\mathbf{G}_\bot$ as the subset $\{\bot\} \subseteq \mathbf{G}$. We denote the first-order (Tarski) entailment relation between subsets of $\mathbf{G}$ by $\models$. This relation satisfies properties (C1)–(C4) of Waldmann et al. [9]).

Note that for *ground* formulas, Tarski entailment (i.e., entailment w.r.t. *all* $\Sigma$-models) agrees with Herbrand entailment (i.e., entailment w.r.t. *term-generated* $\Sigma$-models).

To avoid duplication, we present ordering extensions, selection functions, and inference rules for general clauses, even though we currently need them only for ground clauses.

The term ordering $\succ$ is extended to a literal ordering and a clause ordering in the following way: To every positive literal $s \approx t$, we assign the multiset $\{s, t\}$, to every negative literal $s \not\approx t$, we assign the multiset $\{s, s, t, t\}$. The literal ordering $\succ_{\mathsf{L}}$ compares these multisets using the multiset extension of $\succ$. The clause ordering $\succ_{\mathsf{C}}$ compares clauses by comparing their multisets of literals using the multiset extension of $\succ_{\mathsf{L}}$. We say that a literal $L$ is maximal (strictly maximal) in a clause $C$, if there is no other literal in $C$ that is greater (greater or equal) than $L$ w.r.t. $\succ_{\mathsf{L}}$.

Note that $s \approx t$ and $t \approx s$ are mapped to the same multiset (and analogously for negative literals), so $\succ_{\mathsf{L}}$ is well-defined.

The multiset extension of an ordering that is stable under substitutions is again stable under substitutions, so $\succ_{\mathsf{L}}$ is also stable under substitutions.

A selection function $S$ maps every clause to a submultiset of its negative literals.

For clauses $C$ and $D$ let $\mathrm{rename}(D, C)$ be an arbitrary but fixed renaming substitution $\rho$ such that $D\rho$ and $C$ are variable-disjoint. (In particular, if $C$ and $D$ are variable-disjoint, $\mathrm{rename}(D, C)$ is the identity substitution).

**Inference rules for superposition with ordering $\succ$ and selection function $S$.** Recall that we consider equations as *unordered* pairs of terms, so that all inference rules are to be read modulo symmetry of the equality symbol.

In the non-ground case, we apply a renaming substitution $\rho$ to the first premise of a binary inference to ensure that the premises become variable-disjoint.

*Pos. Superposition:*

$$\frac{D' \vee t \approx t' \qquad C' \vee s[u] \approx s'}{(D'\rho \vee C' \vee s[t'\rho] \approx s')\sigma}$$

where $\rho = \mathrm{rename}(D' \vee t \approx t',\, C' \vee s[u] \approx s')$,
$\sigma = \mathrm{mgu}(t\rho, u)$ and $u$ is not a variable,
$(D' \vee t \approx t')\rho\sigma \not\succeq_{\mathsf{C}} (C' \vee s[u] \approx s')\sigma$,
no literal is selected by $S$ in the premises,
$(t \approx t')\rho\sigma$ is strictly maximal in $(D' \vee t \approx t')\rho\sigma$,
$(s[u] \approx s')\sigma$ is strictly maximal in $(C' \vee s[u] \approx s')\sigma$,
$t\rho\sigma \not\preceq t'\rho\sigma$,
$s[u]\sigma \not\preceq s'\sigma$.

*Neg. Superposition:*

$$\frac{D' \vee t \approx t' \qquad C' \vee s[u] \not\approx s'}{(D'\rho \vee C' \vee s[t'\rho] \not\approx s')\sigma}$$

where $\rho = \mathrm{rename}(D' \vee t \approx t',\, C' \vee s[u] \not\approx s')$,
$\sigma = \mathrm{mgu}(t\rho, u)$ and $u$ is not a variable,
$(D' \vee t \approx t')\rho\sigma \not\succeq_{\mathsf{C}} (C' \vee s[u] \not\approx s')\sigma$,
no literal is selected by $S$ in the left premise,
$(t \approx t')\rho\sigma$ is strictly maximal in $(D' \vee t \approx t')\rho\sigma$,
either $s[u] \not\approx s'$ is selected by $S$ in the right premise
or no literal is selected by $S$ in the right premise
and $(s[u] \not\approx s')\sigma$ is maximal in $(C' \vee s[u] \not\approx s')\sigma$,
$t\rho\sigma \not\preceq t'\rho\sigma$,
$s[u]\sigma \not\preceq s'\sigma$.

*Equality Resolution:*

$$\frac{C' \lor s \not\approx s'}{C'\sigma}$$

where $\sigma = \mathrm{mgu}(s, s')$,
either $s \not\approx s'$ is selected by $S$ in the premise
or no literal is selected by $S$ in the premise
and $(s \not\approx s')\sigma$ is maximal in $(C' \lor s \not\approx s')\sigma$.

*Equality Factoring:*

$$\frac{C' \lor t \approx t' \lor s \approx s'}{(C' \lor s' \not\approx t' \lor s \approx t')\sigma}$$

where $\sigma = \mathrm{mgu}(s, t)$,
no literal is selected by $S$ in the premise,
$(s \approx s')\sigma$ is maximal in $(C' \lor t \approx t' \lor s \approx s')\sigma$,
$s\sigma \not\preceq s'\sigma$.

(These are the most commonly found restrictions for negative superposition and equality resolution, but not the strongest ones. To strengthen the restrictions, one can replace "$L$ is selected by $S$ in $C$" by "$L$ is selected by $S$ in $C$ and $L\sigma$ is maximal in $C^S\sigma$, where $C^S$ is the subclause of $C$ consisting of the literals selected by $S$".)

The **G**-inference system $\mathit{GInf}^{\succ,S}$ consists of all ground inferences $(C_2, C_1, C_0)$ and $(C_1, C_0)$ of the superposition calculus that satisfy the side conditions for $\succ$ and $S$. The formulas $C_n, \dots, C_1$ are called *premises* of an inference $\iota$, $C_0$ is called the *conclusion* of $\iota$, denoted by $\mathit{concl}(\iota)$. If $N \subseteq \mathbf{G}$, we write $\mathit{GInf}^{\succ,S}(N)$ for the set of all inferences in $\mathit{GInf}^{\succ,S}$ whose premises are contained in $N$.

We define the redundancy criterion $\mathit{Red}^{\succ,S} = (\mathit{Red}_{\mathrm{I}}^{\succ,S}, \mathit{Red}_{\mathrm{F}}^{\succ,S})$ for $\mathit{GInf}^{\succ,S}$ as follows:

- Let $N \subseteq \mathbf{G}$. An inference $\iota \in \mathit{GInf}^{\succ,S}$ is contained in $\mathit{Red}_{\mathrm{I}}^{\succ,S}(N)$ if $M \models \mathit{concl}(\iota)$, where $M$ is the set of all clauses in $N$ that are smaller than the right (or only) premise of $\iota$.
- Let $N \subseteq \mathbf{G}$. A clause $C \in \mathbf{G}$ is contained in $\mathit{Red}_{\mathrm{F}}^{\succ,S}(N)$ if $M \models C$, where $M$ is the set of all clauses in $N$ that are smaller than $C$.

By compactness of first-order logic, this is equivalent to "…where $M$ is *some finite* set of clauses in $N$ that are smaller than the right (or only) premise of $\iota$ / that are smaller than $C$".

Inferences in $\mathit{Red}_{\mathrm{I}}^{\succ,S}(N)$ and formulae in $\mathit{Red}_{\mathrm{F}}(N)^{\succ,S}$ are called *redundant w.r.t. $N$*.

$\mathit{Red}^{\succ,S}$ satisfies properties (R1)–(R4) of Waldmann et al. [9]. Since we keep $\succ$ fixed in the sequel, we will usually omit the superscript $\succ$ and write $\mathit{GInf}^S$ and $\mathit{Red}^S$ instead of $\mathit{GInf}^{\succ,S}$ and $\mathit{Red}^{\succ,S}$.

## 4 Ground Refutational Completeness

A set $N \subseteq \mathbf{G}$ is *saturated* w.r.t. $\mathit{GInf}^S$ and $\mathit{Red}^S$ if $\mathit{GInf}^S(N) \subseteq \mathit{Red}^S_{\mathrm{I}}(N)$. The pair $(\mathit{GInf}^S, \mathit{Red}^S)$ is *statically refutationally complete* w.r.t. $\models$ if $\bot \in N$ for every saturated set $N \subseteq \mathbf{G}$ with $N \models \{\bot\}$.

For any set $E$ of ground equations, $\mathcal{T}_\Sigma(\emptyset)/E$ is an $E$-interpretation (or $E$-algebra) with universe $\{\, [t]_E \mid t \in \mathcal{T}_\Sigma(\emptyset) \,\}$, where $[t]_E = \{\, t' \in \mathcal{T}_\Sigma(\emptyset) \mid E \models t \approx t' \,\}$ is the $E$-congruence class of $t \in \mathcal{T}_\Sigma(\emptyset)$.

One can show (similar to the proof of Birkhoff's Theorem) that for every *ground* equation $s \approx t$ we have $E \models s \approx t$ if and only if $\mathcal{T}_\Sigma(\emptyset)/E \models s \approx t$ if and only if $s \leftrightarrow^*_E t$.

In particular, if $E$ is a convergent set of rewrite rules $R$ and $s \approx t$ is a ground equation, then $\mathcal{T}_\Sigma(\emptyset)/R \models s \approx t$ if and only if $s \downarrow_R t$ (i.e., $s \to^*_R u \leftarrow^*_R t$ for some $u$). By abuse of terminology, we say that an equation or clause is valid (or true) in $R$ if and only if it is true in $\mathcal{T}_\Sigma(\emptyset)/R$.

Our refutational completeness proof follows Bachmair and Ganzinger [2, 3]: Given a subset $N \subseteq \mathbf{G}$ with $\bot \notin N$, we first construct a *candidate interpretation*, that is, a convergent set of rewrite rules $R_\infty$. Afterwards we use well-founded induction to show that for a saturated set $N$, $R_\infty$ is actually a model of $N$.

Let $N \subseteq \mathbf{G}$ be a set of clauses not containing $\bot$. Using induction on the clause ordering we define sets of rewrite rules $E_C$ and $R_C$ for all $C \in N$ as follows:

Assume that $E_D$ has already been defined for all $D \in N$ with $D \prec_{\mathsf{C}} C$. Then $R_C = \bigcup_{D \prec_{\mathsf{C}} C} E_D$. The set $E_C$ contains the rewrite rule $s \to t$, if

(a) $C = C' \vee s \approx t$.
(b) no literal is selected in $C$.
(c) $s \approx t$ is strictly maximal in $C$.
(d) $s \succ t$.
(e) $C$ is false in $R_C$.
(f) $C'$ is false in $R_C \cup \{s \to t\}$.
(g) $s$ is irreducible w.r.t. $R_C$.

In this case, $C$ is called *productive*. Otherwise $E_C = \emptyset$. Finally, we define $R_\infty = \bigcup_{D \in N} E_D$.

**Lemma 1.** *If $E_C = \{s \to t\}$ and $E_D = \{u \to v\}$, then $s \succ u$ if and only if $C \succ_{\mathsf{C}} D$.*

*Proof.* ($\Rightarrow$): By condition (c), $s \approx t$ is strictly maximal in $C$ and $u \approx v$ is strictly maximal in $D$, and since the literal ordering is total on ground literals, this implies that all other literals in $C$ or in $D$ are actually smaller than $s \approx t$ or $u \approx v$, respectively.

Moreover, $s \succ t$ and $u \succ v$ by condition (d). Therefore $s \succ u$ implies $\{s, t\} \succ_{\mathsf{mul}} \{u, v\}$. Hence $s \approx t \succ_{\mathsf{L}} u \approx v \succeq_{\mathsf{L}} L$ for every literal $L$ of $D$, and thus $C \succ_{\mathsf{C}} D$.

($\Leftarrow$): Let $C \succ_{\mathsf{C}} D$, then $E_D \subseteq R_C$. By condition (g), $s$ must be irreducible w.r.t. $R_C$, so $s \neq u$.

Assume that $s \not\succ u$. By totality, this implies $s \preceq u$, and since $s \neq u$, we obtain $s \prec u$. But then $C \prec_{\mathsf{C}} D$ can be shown in the same way as in the ($\Rightarrow$)-part, contradicting the assumption. □

**Corollary 2.** *The rewrite systems $R_C$ and $R_\infty$ are convergent (i.e., terminating and confluent).*

*Proof.* By condition (d), $s \succ t$ for all rules $s \to t$ in $R_C$ and $R_\infty$, so $R_C$ and $R_\infty$ are terminating.

Furthermore, it is easy to check that there are no critical pairs between any two rules: Assume that there are rules $u \to v$ in $E_D$ and $s \to t$ in $E_C$ such that $u$ is a subterm of $s$. As $\succ$ is a reduction ordering that is total on ground terms, we get $u \prec s$ and therefore $D \prec_{\mathsf{C}} C$ and $E_D \subseteq R_C$. But then $s$ would be reducible by $R_C$, contradicting condition (g).

Now the absence of critical pairs implies local confluence, and termination and local confluence imply confluence. □

**Lemma 3.** *If $D \preceq_{\mathsf{C}} C$ and $E_C = \{s \to t\}$, then $s \succ u$ for every term $u$ occurring in a negative literal in $D$ and $s \succeq u$ for every term $u$ occurring in a positive literal in $D$.*

*Proof.* If $s \preceq u$ for some term $u$ occurring in a negative literal $u \not\approx v$ in $D$, then $\{u, u, v, v\} \succ_{\mathsf{mul}} \{s, t\}$. So $u \not\approx v \succ_{\mathsf{L}} s \approx t \succeq_{\mathsf{L}} L$ for every literal $L$ of $C$, and therefore $D \succ_{\mathsf{C}} C$.

Similarly, if $s \prec u$ for some term $u$ occurring in a positive literal $u \approx v$ in $D$, then $\{u, v\} \succ_{\mathsf{mul}} \{s, t\}$. So $u \approx v \succ_{\mathsf{L}} s \approx t \succeq_{\mathsf{L}} L$ for every literal $L$ of $C$, and therefore $D \succ_{\mathsf{C}} C$. □

**Corollary 4.** *If $D \in N$ is true in $R_D$, then $D$ is true in $R_\infty$ and $R_C$ for all $C \succ_{\mathsf{C}} D$.*

*Proof.* If a positive literal $s \approx t$ of $D$ is true in $R_D$, then $s \downarrow_{R_D} t$. Since $R_D \subseteq R_C$ and $R_D \subseteq R_\infty$, we have $s \downarrow_{R_C} t$ and $s \downarrow_{R_\infty} t$, so $s \approx t$ is true in $R_C$ and $R_\infty$.

Otherwise, some negative literal $s \not\approx t$ of $D$ must be true in $R_D$, hence $s \not\downarrow_{R_D} t$. As the rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than $s$ and $t$, they cannot be used in a rewrite proof of $s \downarrow t$, hence $s \not\downarrow_{R_C} t$ and $s \not\downarrow_{R_\infty} t$. □

**Corollary 5.** *If $D = D' \vee u \approx v$ is productive, then $D'$ is false and $D$ is true in $R_\infty$ and $R_C$ for all $C \succ_{\mathsf{C}} D$.*

*Proof.* Obviously, $D$ is true in $R_\infty$ and $R_C$ for all $C \succ_{\mathsf{C}} D$.

Since all negative literals of $D'$ are false in $R_D$, it is clear that they are false in $R_\infty$ and $R_C$. For the positive literals $u' \approx v'$ of $D'$, condition (f) ensures that they are false in $R_D \cup \{u \to v\}$. Since $u' \preceq u$ and $v' \preceq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than $u$, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$. □

**Theorem 6 ("Model Construction").** *Let $N$ be a set of clauses that is saturated and does not contain the empty clause. Then we have for every ground clause $C \in N$:*

*(i) $E_C = \emptyset$ if and only if $C$ is true in $R_C$.*

*(ii) $C$ is true in $R_\infty$ and in $R_D$ for every $D \in N$ with $D \succ_{\mathsf{C}} C$.*

*Proof.* We use induction on the clause ordering $\succ_{\mathsf{C}}$ and assume that (i) and (ii) are already satisfied for all clauses in $N$ that are smaller than $C$. Note that the "if" part of (i) is obvious from the construction and that condition (ii) follows immediately from (i) and Corollaries 4 and 5. So it remains to show the "only if" part of (i).

Case 1: *$C$ contains selected literals or a maximal negative literal.* Suppose that $C = C' \vee s \not\approx s'$, where $s \not\approx s'$ is a selected literal of $C$ if $C$ has selected literal, and where $s \not\approx s'$ is a maximal literal of $C$ if $C$ does not have selected literals. If $s \approx s'$ is false in $R_C$, then $C$ is clearly true in $R_C$ and we are done. So assume that $s \approx s'$ is true in $R_C$, that is, $s \downarrow_{R_C} s'$. Without loss of generality, $s \succeq s'$.

Case 1.1: $s = s'$. If $s = s'$, then there is an *equality resolution* inference

$$\frac{C' \vee s \not\approx s'}{C'}\ .$$

As $N$ is saturated, this inference is contained in $Red_{\mathrm{I}}^{S}(N)$. So $M \models C'$, where $M$ is the set of all clauses in $N$ that are smaller than $C$. By the induction hypothesis, all clauses in $M$ are true in $R_C$, therefore $C'$ and $C$ are true in $R_C$.

Case 1.2: $s \succ s'$. By definition, $s \downarrow_{R_C} s'$ means $s \rightarrow^*_{R_C} u \leftarrow^*_{R_C} s'$ for some term $u$. Since $s' \rightarrow^*_{R_C} u$, we know that $s' \succeq u$. If $s \succ s'$, then the derivation $s \rightarrow^*_{R_C} u$ cannot be empty, so it has the form $s = s[t] \rightarrow_{R_C} s[t'] \rightarrow^*_{R_C} u$, where $t \rightarrow t'$ is a rule in $E_D \subseteq R_C$ for some $D \in N$ with $D \prec_{\mathsf{C}} C$. Let $D = D' \vee t \approx t'$ with $E_D = \{t \rightarrow t'\}$. By property (b), no literal in $D$ may be selected. Consequently, there is a *negative superposition* inference

$$\frac{D' \vee t \approx t' \qquad C' \vee s[t] \not\approx s'}{D' \vee C' \vee s[t'] \not\approx s'}$$

from $D$ and $C$. As $N$ is saturated, this inference is contained in $Red_{\mathrm{I}}^{S}(N)$. So its conclusion $D' \vee C' \vee s[t'] \not\approx s'$ is entailed by the set $M$ of all clauses in $N$ that are smaller than $C$. By the induction hypothesis, all clauses in $M$ are true in $R_C$, therefore the conclusion is true in $R_C$. Since $D$ is productive, $D'$ is false in $R_C$ by Cor. 5. Moreover, $s[t'] \rightarrow^*_{R_C} u \leftarrow^*_{R_C} s'$, so $s[t'] \not\approx s'$ is also false in $R_C$. Since $D'$ and $s[t'] \not\approx s'$ are false in $R_C$, $C'$ must be true, and therefore $C$ is also true in $R_C$.

Case 2: *$C$ contains neither selected literals nor a maximal negative literal.* If $C$ does not fall into Case 1, it must have the form $C' \vee s \approx s'$, where $s \approx s'$ is a maximal literal of $C$. If $E_C = \{s \rightarrow s'\}$ or $C'$ is true in $R_C$ or $s = s'$, then there is nothing to show, so assume that $E_C = \emptyset$ and that $C'$ is false in $R_C$. Without loss of generality, $s \succ s'$.

Case 2.1: *$s \approx s'$ is maximal in $C$, but not strictly maximal.* If $s \approx s'$ is maximal in $C$, but not strictly maximal, then $C$ can be written as $C'' \vee s \approx s' \vee s \approx s'$. In this case, there is a *equality factoring* inference

$$\frac{C'' \vee s \approx s' \vee s \approx s'}{C'' \vee s' \not\approx s' \vee s \approx s'}$$

As in Case 1, saturation implies that the conclusion is true in $R_C$. Since $s' = s'$ implies $s' \downarrow_{R_C} s'$, we know that $s' \not\approx s'$ is false in $R_C$. So $C'' \vee s \approx s'$ must be true, and therefore $C$ is also true in $R_C$.

Case 2.2: *$s \approx s'$ is strictly maximal in $C$ and $s$ is reducible.* Suppose that $s \approx s'$ is strictly maximal in $C$ and $s$ is reducible by some rule in $E_D \subseteq R_C$. Let $D = D' \vee t \approx t'$ and $E_D = \{t \to t'\}$. By property (b), no literal in $D$ may be selected. Consequently, there is a *positive superposition* inference

$$\frac{D' \vee t \approx t' \qquad C' \vee s[t] \approx s'}{D' \vee C' \vee s[t'] \approx s'}$$

from $D$ and $C$. Again, saturation implies that the conclusion is true in $R_C$. Since $D$ is productive, $D'$ is false in $R_C$ by Cor. 5. Since $D'$ and $C'$ are false in $R_C$, $s[t'] \approx s'$ must be true in $R_C$, that is, $s[t'] \to^*_{R_C} u \leftarrow^*_{R_C} s'$. On the other hand, $s[t] \to_{R_C} s[t']$, so $s[t] \downarrow_{R_C} s'$, which means that $s[t] \approx s'$ and $C$ are true in $R_C$.

Case 2.3: *$s \approx s'$ is strictly maximal in $C$ and $s$ is irreducible.* Suppose that $s \approx s'$ is strictly maximal in $C$ and $s$ is irreducible by $R_C$. Then conditions (a)–(d) and (g) for productivity are satisfied. If $C$ is productive, there is nothing to show. If $C$ is not productive, then either property (e) or (f) must be violated. If (e) is violated, that is, if $C$ is true in $R_C$, there is again nothing to show. Let us therefore assume that (e) holds but (f) does not hold, that is, $C$ (and hence $C'$) is false in $R_C$ but $C'$ is true in $R_C \cup \{s \to s'\}$. Clearly any negative literal that is true in $R_C \cup \{s \to s'\}$ is also true in $R_C$. So $C'$ must have the form $C' = C'' \vee t \approx t'$, where the positive literal $t \approx t'$ is true in $R_C \cup \{s \to s'\}$ and false in $R_C$. In other words, $t \downarrow_{R_C \cup \{s \to s'\}} t'$, but not $t \downarrow_{R_C} t'$. Consequently, there is a rewrite proof of $t \to^* u \leftarrow^* t'$ by $R_C \cup \{s \to s'\}$ in which the rule $s \to s'$ is used at least once. Without loss of generality we assume that $t \succeq t'$. If $t$ were strictly smaller than $s$, it would be impossible to use $s \to s'$ in the rewrite proof. If $t$ were strictly larger than $s$ (which is again larger than $s'$), then $s \approx s' \prec_{\mathsf{L}} t \approx t'$, contradicting the assumption that $s \approx s'$ is strictly maximal in $C$. So we have $s = t$. Moreover, since $s \approx s'$ is strictly maximal in $C$, we must have $t = s \succ s' \succ t'$. From $t \succ t'$ we conclude that the rewrite proof of $t \to^* u \leftarrow^* t'$ has the form $t \to t'' \to^* u \leftarrow^* t'$. Since $t \succ t''$ and $t \succ t'$, every left-hand side of a rule used in $t'' \to^* u \leftarrow^* t'$ must be strictly smaller than $t$. Because $s \to s'$ must be used at least once in $t \to t'' \to^* u \leftarrow^* t'$ and cannot be used after the first step, the rewrite proof has the form $t = s \to s' \to^* u \leftarrow^* t'$, where the first step uses $s \to s'$ and all other steps use rules from $R_C$. Consequently, $s' \approx t'$ is true in $R_C$. Now observe that there is an *equality factoring* inference

$$\frac{C'' \vee t \approx t' \vee s \approx s'}{C'' \vee s' \not\approx t' \vee t \approx t'}$$

whose conclusion is true in $R_C$ by saturation. Since the literal $s' \not\approx t'$ must be false in $R_C$, the rest of the clause must be true in $R_C$, and therefore $C$ must be true in $R_C$, contradicting our assumption. This concludes the proof of the theorem. □

**Corollary 7 ("Static Refutational Completeness").** *The pair* $(\mathit{GInf}^S, \mathit{Red}^S)$ *is statically refutationally complete w.r.t.* $\models$ *for every selection function* $S$.

*Proof.* Let $N$ be a subset of $\mathbf{G}$ that does not contain $\bot$. By part (ii) of the model construction theorem, the interpretation $R_\infty$ (that is, $\mathcal{T}_\Sigma(\emptyset)/R_\infty$) is a model of all clauses in $N$. □

## 5 Lifting

We will now lift the completeness result for ground first-order clauses to a completeness result for general first-order clauses.

Let $\mathbf{F}$ be the set of first-order clauses over $\Sigma$; let $\mathbf{F}_\bot$ be the subset $\{\bot\} \subseteq \mathbf{F}$. The $\mathbf{F}$-inference system $\mathit{FInf}^{\succ,S}$ consists of all inferences $(C_2, C_1, C_0)$ and $(C_1, C_0)$ of the superposition calculus that satisfy the side conditions for $\succ$ and $S$. The formulas $C_n, \dots, C_1$ are called *premises* of an inference $\iota$, $C_0$ is called the *conclusion* of $\iota$, denoted by $\mathit{concl}(\iota)$. If $N \subseteq \mathbf{F}$, we write $\mathit{FInf}^{\succ,S}(N)$ for the set of all inferences in $\mathit{FInf}^{\succ,S}$ whose premises are contained in $N$.

A selection function $T$ for $\mathbf{G}$ is a grounding of a selection function $S$ for $\mathbf{F}$, if for every $D \in \mathbf{G}$ there exists some $C \in \mathbf{F}$ such that $D = C\theta$ and $T(D) = (S(C))\theta$, that is, if for every clause $D \in \mathbf{G}$ the literals selected by $T$ in $D$ correspond to the literals selected by $S$ in some $C \in \mathbf{F}$ such that $D = C\theta$. The set of all groundings of a selection function $S$ is denoted by $\mathrm{gs}(S)$.

If $\iota = (C_1, C_0)$ is an inference in $\mathit{FInf}^{\succ,S}$ and $\theta$ is a substitution such that $C_1\theta, C_0\theta \in \mathbf{G}$, then $(C_1\theta, C_0\theta)$ is called a pre-instance of $\iota$. Analogously, if $\iota = (C_2, C_1, C_0)$ is an inference in $\mathit{FInf}^{\succ,S}$, $\rho = \mathrm{rename}(C_2, C_1)$, and $\theta$ is a substitution such that $C_2\rho\theta, C_1\theta, C_0\theta \in \mathbf{G}$, then $(C_2\rho\theta, C_1\theta, C_0\theta)$ is called a pre-instance of $\iota$.

Let $\iota$ be an inference $(C_1, C_0)$ or $(C_2, C_1, C_0)$ in $\mathit{FInf}^{\succ,S}$, let $\iota'$ be a pre-instance $(C_1\theta, C_0\theta)$ or $(C_2\rho\theta, C_1\theta, C_0\theta)$ of $\iota$, and let $T \in \mathrm{gs}(S)$. Then $\iota'$ is called a $T$-ground instance of $\iota$ if it is a $\mathbf{G}$-inference in $\mathit{GInf}^{\succ,T}$ and $T(C_1\theta) = (S(C_1))\theta$ (and $T(C_2\rho\theta) = (S(C_2))\rho\theta$).

Note that the definition of a $T$-ground instance depends implicitly on $S$. In fact, it is possible that $\iota$ is an inference in both $\mathit{FInf}^{\succ,S_1}$ and $\mathit{FInf}^{\succ,S_2}$ and $T \in \mathrm{gs}(S_1) \cap \mathrm{gs}(S_2)$, and that the pre-instance $\iota'$ is a $T$-ground instance of $\iota$, if we consider $\iota$ as an $\mathit{FInf}^{\succ,S_1}$ inference, but not, if we consider $\iota$ as an $\mathit{FInf}^{\succ,S_2}$ inference. In a formalized proof, one should rather talk about $(S, T)$-ground instances instead of $T$-ground instances.

Let $S$ be a selection function for $\mathbf{F}$, Let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$. We define the grounding function $\mathcal{G}^T$ as follows:

- For a clause $C \in \mathbf{F}$, $\mathcal{G}^T(C) = \{\, C\theta \mid C\theta \in \mathbf{G} \,\}$ is the set of all ground instances of $C$.

- For an **F**-inference $\iota \in \mathit{FInf}^{\succ,S}$, $\mathcal{G}^T(\iota)$ is the set of all $T$-ground instances of $\iota$.

The function $\mathcal{G}$ is extended to sets $\mathcal{S}$ of formulas or inferences by defining $\mathcal{G}(\mathcal{S}) = \bigcup_{x\in\mathcal{S}} \mathcal{G}(x)$.

$\mathcal{G}^T$ satisfies properties (G1)–(G3) of grounding functions of Waldmann et al. [9]; in fact, it also satisfies (G3′), which implies (G3).

For $T \in \mathrm{gs}(S)$, the $\mathcal{G}^T$-*lifting* of $\models$ is the relation $\models^{\mathcal{G}^T} \subseteq \mathcal{P}(\mathbf{F}) \times \mathcal{P}(\mathbf{F})$ defined by $N_1 \models^{\mathcal{G}^T} N_2$ if and only if $\mathcal{G}^T(N_1) \models \mathcal{G}^T(N_2)$. Note that for sets $N$ of clauses, $\mathcal{G}^T(N)$ is independent of $T$, which implies that $\models^{\mathcal{G}^T}$ is also independent of $T$. In fact, $\models^{\mathcal{G}^T}$ is the Herbrand entailment relation $\models_{\mathsf{H}}$ for sets of first-order clauses (which in turn is equivalent to the standard (Tarskian) entailment relation $\models_{\mathsf{T}}$ for sets of first-order clauses as long as we are only interested in refutations, i.e., $N \models_{\mathsf{H}} \bot$ holds if and only if $N \models_{\mathsf{T}} \bot$ holds). Trivially, the intersection $\models^{\cap}$ of all $\models^{\mathcal{G}^T}$ for $T \in \mathrm{gs}(S)$ is again $\models_{\mathsf{H}}$.

We define the $\mathcal{G}^T$-*lifting* $\mathit{Red}^{\mathcal{G}^T} = (\mathit{Red}_{\mathrm{I}}^{\mathcal{G}^T}, \mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T})$ of $\mathit{Red}^T$ with $\mathit{Red}_{\mathrm{I}}^{\mathcal{G}^T} : \mathcal{P}(\mathbf{F}) \to \mathcal{P}(\mathit{FInf}^{\succ,S})$ and $\mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T} : \mathcal{P}(\mathbf{F}) \to \mathcal{P}(\mathbf{F})$ by $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^T}(N)$ if and only if $\mathcal{G}^T(\iota) \subseteq \mathit{Red}_{\mathrm{I}}^T(\mathcal{G}^T(N))$ and $C \in \mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T}(N)$ if and only if $\mathcal{G}^T(C) \subseteq \mathit{Red}_{\mathrm{F}}^T(\mathcal{G}^T(N))$.

For every $T \in \mathrm{gs}(S)$, $\mathit{Red}^{\mathcal{G}^T}$ is a redundancy criterion for $\mathit{FInf}^{\succ,S}$ and $\models_{\mathsf{H}}$ by Thm. 30 of Waldmann et al. [9]. Moreover, the intersection $\mathit{Red}^{\cap} = \bigcap_{T\in\mathrm{gs}(S)} \mathit{Red}^{\mathcal{G}^T}$ is a redundancy criterion for $\mathit{FInf}^{\succ,S}$ and $\models_{\mathsf{H}}$ by Thm. 24 of [9].

**Lemma 8.** *Let $C = C' \vee s \not\approx s' \in \mathbf{F}$, let $\theta$ be a substitution such that $C\theta \in \mathbf{G}$. Let $S$ be a selection function for $\mathbf{F}$, let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$ such that $T(C\theta) = (S(C))\theta$. Let*

$$\iota = \frac{C'\theta \vee s\theta \not\approx s'\theta}{C'\theta}$$

*with $s\theta = s'\theta$ be an equality resolution inference in $\mathit{GInf}^{\succ,T}$ from $C\theta$. Then $\iota$ is a $T$-ground instance of an equality resolution inference in $\mathit{FInf}^{\succ,S}$ from $C$.*

*Proof.* Since $s\theta = s'\theta$, the terms $s$ and $s'$ are unifiable. Let $\sigma$ be an idempotent most general unifier of $s$ and $s'$ such that $\theta = \sigma \circ \tau$. Note that by idempotence, $\sigma \circ \theta = \sigma \circ \sigma \circ \tau = \sigma \circ \tau = \theta$.

Suppose that $\iota$ is an equality resolution inference in $\mathit{GInf}^{\succ,T}$ from $C\theta$. Then either $s\theta \not\approx s'\theta$ is selected in $C\theta$ by $T$ or no literal is selected in $C\theta$ by $T$ and $s\theta \not\approx s'\theta$ is maximal in $C\theta$. Since $T(C\theta) = (S(C))\theta$, in the first case $s \not\approx s'$ is selected in $C$ by $S$, and in the second case no literal is selected in $C$ by $S$. Moreover, in the second case, $s\sigma \not\approx s'\sigma$ must be a maximal literal in $C\sigma$ (if it were not maximal, then $L\sigma \succ_{\mathsf{L}} s\sigma \not\approx s'\sigma$ for some other literal $L\sigma$ in $C\sigma$, hence $L\theta = L\sigma\tau \succ_{\mathsf{L}} s\sigma\tau \not\approx s'\sigma\tau = s\theta \not\approx s'\theta$ for a literal $L\theta$ in $C\theta$, contradicting the maximality of $s\theta \not\approx s'\theta$). Therefore

$$\iota' = \frac{C' \vee s \not\approx s'}{C'\sigma}$$

is an equality resolution inference in $FInf^{\succ,S}$ from $C$. Moreover $C'\sigma\theta = C'\theta$, and by assumption $T(C\theta) = (S(C))\theta$, so $\iota \in \mathcal{G}^T(\iota')$. ☐

**Lemma 9.** *Let $C = C' \vee t \approx t' \vee s \approx s' \in \mathbf{F}$, let $\theta$ be a substitution such that $C\theta \in \mathbf{G}$. Let $S$ be a selection function for $\mathbf{F}$, let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$ such that $T(C\theta) = (S(C))\theta$. Let*

$$\iota = \frac{C'\theta \vee t\theta \approx t'\theta \vee s\theta \approx s'\theta}{C'\theta \vee s'\theta \not\approx t'\theta \vee s\theta \approx t'\theta}$$

*with $s\theta = t\theta$ be an equality factoring inference in $GInf^{\succ,T}$ from $C\theta$. Then $\iota$ is a $T$-ground instance of an equality factoring inference in $FInf^{\succ,S}$ from $C$.*

*Proof.* Since $s\theta = t\theta$, the terms $s$ and $t$ are unifiable. Let $\sigma$ be an idempotent most general unifier of $s$ and $t$ such that $\theta = \sigma \circ \tau$. Note that by idempotence, $\sigma \circ \theta = \sigma \circ \sigma \circ \tau = \sigma \circ \tau = \theta$.

Suppose that $\iota$ is an equality factoring inference in $GInf^{\succ,T}$ from $C\theta$. Then no literal is selected in $C\theta$ by $T$, and since $T(C\theta) = (S(C))\theta$, this implies that no literal is selected in $C$ by $S$. Furthermore $s\theta \approx s'\theta$ must be a maximal literal in $C\theta$, which implies that $s\sigma \approx s'\sigma$ must be a maximal literal in $C\sigma$. Finally, we know that $s\theta \succ s'\theta$, hence $s\sigma \not\preceq s'\sigma$ (since $s\sigma \preceq s'\sigma$ would imply $s\theta = s\sigma\tau \preceq s'\sigma\tau = s'\theta$). Therefore

$$\iota' = \frac{C' \vee t \approx t' \vee s \approx s'}{(C' \vee s' \not\approx t' \vee s \approx t')\sigma}$$

is an equality factoring inference in $FInf^{\succ,S}$ from $C$. Moreover $(C' \vee s' \not\approx t' \vee s \approx t')\sigma\theta = (C' \vee s' \not\approx t' \vee s \approx t')\theta$, and by assumption $T(C\theta) = (S(C))\theta$, so $\iota \in \mathcal{G}^T(\iota')$. ☐

**Lemma 10.** *Let $D = D' \vee t \approx t'$ and $C = C' \vee [\neg]\, s \approx s'$ be two clauses in $\mathbf{F}$; let $\theta_1$ and $\theta_2$ be substitutions such that $D\theta_2 \in \mathbf{G}$ and $C\theta_1 \in \mathbf{G}$. Let $S$ be a selection function for $\mathbf{F}$, let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$ such that $T(C\theta_1) = (S(C))\theta_1$ and $T(D\theta_2) = (S(D))\theta_2$. Let*

$$\iota = \frac{D'\theta_2 \vee t\theta_2 \approx t'\theta_2 \qquad C'\theta_1 \vee [\neg]\, s\theta_1[v]_p \approx s'\theta_1}{D'\theta_2 \vee C'\theta_1 \vee [\neg]\, s\theta_1[t'\theta_2]_p \approx s'\theta_1}$$

*with $t\theta_2 = v = s\theta_1|_p$ be a positive or negative superposition inference in $GInf^{\succ,T}$ from $D\theta_2$ and $C\theta_1$. If $p$ is a position of $s$ and $s|_p$ is not a variable, then $\iota$ is a $T$-ground instance of a superposition inference in $FInf^{\succ,S}$ from $D$ and $C$.*

*Proof.* We consider the case of negative superposition inferences, the proof for positive superposition inferences is similar.

Let $\rho = \mathrm{rename}(D, C)$, then $D\rho$ and $C$ are variable-disjoint. Define the substitution $\theta$ by $x\theta = x\theta_1$ if $x$ is a variable of $C$ and $x\theta = x\rho^{-1}\theta_2$ if $x$ is a variable of $D\rho$. Clearly $C\theta = C\theta_1$ and $D\rho\theta = D\rho\rho^{-1}\theta_2 = D\theta_2$.

Assume that $p$ is a position of $s$ and that $s|_p$ is not a variable. Let $u = s|_p$. Then $t\rho\theta = t\theta_2 = v = u\theta$ and we have $s\theta = s\theta[v]_p = s\theta[u\theta]_p = (s[u]_p)\theta$ and $s\theta[t'\rho\theta]_p = (s[t'\rho]_p)\theta$. Since $t\rho\theta = u\theta$, the terms $t\rho$ and $u$ are unifiable. Let $\sigma$ be an idempotent most general unifier of $t\rho$ and $u$ such that $\theta = \sigma \circ \tau$. Note that by idempotence, $\sigma \circ \theta = \sigma \circ \sigma \circ \tau = \sigma \circ \tau = \theta$.

Suppose that $\iota$ is a negative resolution inference in $\mathit{GInf}^{\succ,T}$ from $D\theta$ and $C\theta$. Then either $s\theta \not\approx s'\theta$ is selected in $C\theta$ by $T$ or no literal is selected in $C\theta$ by $T$ and $s\theta \not\approx s'\theta$ is maximal in $C\theta$. Since $T(C\theta) = (S(C))\theta$, in the first case $s \not\approx s'$ is selected in $C$ by $S$, and in the second case no literal is selected in $C$ by $S$, and moreover, in the second case, $s\sigma \not\approx s'\sigma$ must be a maximal literal in $C\sigma$.

Similarly, no literal may be selected in $D\rho\theta$ by $T$ and $t\rho\theta \approx t'\rho\theta$ must be strictly maximal in $D\rho\theta$. Since $T(D\rho\theta) = (S(D))\rho\theta$, this implies that zno literal is selected in $D$ by $S$ and that $t\rho\sigma \approx t'\rho\sigma$ must be a strictly maximal literal in $D\rho\sigma$.

Finally, $t\rho\theta \succ t'\rho\theta$, $s[u]\theta \succ s'\theta$, and $D\rho\theta \not\succeq_{\mathsf{C}} C\theta$, from which we conclude that $t\rho\sigma \not\preceq t'\rho\sigma$, $s[u]\sigma \not\preceq s'\sigma$, and $D\rho\sigma \not\succeq_{\mathsf{C}} C\sigma$.

Therefore

$$\iota' = \frac{D' \vee t \approx t' \qquad C' \vee s[u]_p \not\approx s'}{(D'\rho \vee C' \vee s[t'\rho]_p \not\approx s')\sigma}$$

is a negative superposition inference in $\mathit{FInf}^{\succ,S}$ from $D$ and $C$. Moreover $D\rho\theta = D\theta_2$ and thus $T(D\rho\theta) = T(D\theta_2) = (S(D))\theta_2 = (S(D))\rho\theta$, $C\theta = C\theta_1$ and thus $T(C\theta) = T(C\theta_1) = (S(C))\theta_1 = (S(C))\theta$, and $(D'\rho \vee C' \vee s[t'\rho] \not\approx s')\sigma\theta = (D'\rho \vee C' \vee s[t'\rho] \not\approx s')\theta = (D'\theta_2 \vee C'\theta_1 \vee s\theta_1[t'\theta_2] \not\approx s'\theta_1)$, so $\iota \in \mathcal{G}^T(\iota')$. □

**Lemma 11.** *Let $N \subseteq \mathbf{F}$. Let $S$ be a selection function for $\mathbf{F}$ and let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$, such that for every $D \in \mathcal{G}^T(N)$ there exists some $C \in N$ and a some substitution $\theta$ such that $D = C\theta$ and $T(D) = (S(C))\theta$. Then $\mathit{GInf}^{\succ,T}(\mathcal{G}^T(N)) \subseteq \mathcal{G}^T(\mathit{FInf}^{\succ,S}(N)) \cup \mathit{Red}_{\mathrm{I}}^T(\mathcal{G}^T(N))$.*

*Proof.* Let $\iota$ be an inference in $\mathit{GInf}^{\succ,T}$ from premises $D_n, \dots, D_1 \in \mathcal{G}^T(N)$ (with $n \in \{1,2\}$). By assumption, there exist $C_n, \dots, C_1 \in N$ and substitutions $\theta_i$ such that $D_i = C_i\theta_i$ and $T(D_i) = (S(C_i))\theta_i$.

If $\iota$ is an equality resolution inference with premise $D_1 = D' \vee v \not\approx v'$, then $C_1$ must have the form $C' \vee s \not\approx s'$ with $C'\theta_1 = D'$, $s\theta_1 = v$, and $s'\theta_1 = v'$. By Lemma 8, $\iota \in \mathcal{G}^T(\iota')$ for some $\iota' \in \mathit{FInf}^{\succ,S}(N)$.

If $\iota$ is an equality factoring inference with premise $D_1 = D' \vee v \approx v' \vee v \approx v''$, then $C_1$ must have the form $C' \vee t \approx t' \vee s \approx s'$ with $C'\theta_1 = D'$, $s\theta_1 = t\theta_1 = v$, $t'\theta_1 = v'$, and $s'\theta_1 = v''$. By Lemma 9, $\iota \in \mathcal{G}^T(\iota')$ for some $\iota' \in \mathit{FInf}^{\succ,S}(N)$.

Otherwise, $\iota$ is a positive or negative superposition inference with premises $D_2 = D_2' \vee v \approx v'$ and $D_1 = D_1' \vee [\neg]\, u[v]_p \approx u'$. Then $C_2$ and $C_1$ must have the form $C_2 = C_2' \vee t \approx t'$ and $C_1 = C_1' \vee [\neg]\, s \approx s'$ with $C_2'\theta_2 = D_2'$, $t\theta_2 = v$, $t'\theta_2 = v'$, and $C_1'\theta_1 = D_1'$, $s\theta_1 = u[v]_p$, $s'\theta_1 = u'$.

If $p$ is a position of $s$ and $s|_p$ is not a variable, then $\iota \in \mathcal{G}^T(\iota')$ for some $\iota' \in \mathit{FInf}^{\succ,S}(N)$ by Lemma 10.

Otherwise let $p'$ be the longest prefix of $p$ that is a position of $s$ and let $p = p'p''$. The subterm $s|_{p'}$ must be a variable $x$. Define the substitution $\theta_1'$ by

$x\theta_1' = x\theta_1[v']_{p''}$ and $y\theta_1' = y\theta_1$ for every variable $y$ of $C_1$ different from $x$. Clearly, $C_1\theta_1' \in \mathcal{G}^T(N)$, and since $v \succ v'$ we have $x\theta_1 = x\theta_1[v]_{p''} \succ x\theta_1[v']_{p''} = x\theta_1'$ and thus $D_1 = C_1\theta_1 \succ_{\mathsf{C}} C_1\theta_1'$. Furthermore, we already know from the definition of superposition inferences that $D_1 \succ_{\mathsf{C}} D_2$. We will show that $C_1\theta_1'$ and $D_2$ entail the conclusion $D_2' \vee D_1' \vee [\neg]\, u[v']_p \approx u'$ of $\iota$: Suppose that $C_1\theta_1'$ and $D_2 = D_2' \vee v \approx v'$ hold in some interpretation. If $D_2'$ holds, then the conclusion holds trivially. Otherwise, $v \approx v'$ holds in that interpretation, then $x\theta_1[v]_{p''} \approx x\theta_1[v']_{p''}$ holds by congruence, and since $x\theta_1 = x\theta[v]_{p''}$ and $x\theta_1' = x\theta_1[v']_{p''}$, we know that $x\theta_1 \approx x\theta_1'$ holds. Moreover, $y\theta_1 \approx y\theta_1'$ holds for every variable $y$ different from $x$. Since $C_1\theta_1'$ holds in the interpretation by assumption, congruence implies that $C_1\theta_1 = D_1' \vee [\neg]\, u[v]_p \approx u'$ holds in the interpretation. Once more by congruence, $D_1' \vee [\neg]\, u[v']_p \approx u'$ holds, so the conclusion of $\iota$ must hold as well. Therefore, the conclusion of $\iota$ is entailed by $C_1\theta_1'$ and $D_2$; both clauses are contained in $\mathcal{G}^T(N)$ and smaller than the right premise of $\iota$; so $\iota \in \mathit{Red}_{\mathrm{I}}^T(\mathcal{G}^T(N))$. □

By Lemma 31 of Waldmann et al. [9], we get immediately:

**Lemma 12.** *Let $N \subseteq \mathbf{F}$. Let $S$ be a selection function for $\mathbf{F}$ and let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$, such that for every $D \in \mathcal{G}^T(N)$ there exists some $C \in N$ and a some substitution $\theta$ such that $D = C\theta$ and $T(D) = (S(C))\theta$. If $N$ is saturated w.r.t. $\mathit{FInf}^{\succ,S}$ and $\mathit{Red}^{\mathcal{G}^T}$, then $\mathcal{G}^T(N)$ is saturated w.r.t. $\mathit{GInf}^{\succ,T}$ and $\mathit{Red}^T$.*

From this, static refutational completeness of $(\mathit{FInf}^{\succ,S}, \mathit{Red}^{\cap})$ follows by choosing an appropriate selection function $T \in \mathrm{gs}(S)$ for a given saturated set $N$:

**Theorem 13.** *$(\mathit{FInf}^{\succ,S}, \mathit{Red}^{\cap})$ is statically refutationally complete.*

*Proof.* Let $N \subseteq \mathbf{F}$ be saturated w.r.t. $\mathit{FInf}^{\succ,S}$ and $\mathit{Red}^{\cap}$. Suppose that $\bot \notin N$. We define a selection function $T \in \mathrm{gs}(S)$ as follows: If there exists a $C \in N$ such that $D = C\theta$ for some $\theta$, we set $T(D) = (S(C))\theta$ for some $C \in N$ with this property; otherwise we set $T(D) = (S(C))\theta$ for an arbitrary $C \in \mathbf{F}$ such that $D = C\theta$. Since $N$ is saturated w.r.t. $\mathit{FInf}^{\succ,S}$ and $\mathit{Red}^{\cap}$, $N$ is saturated w.r.t. $\mathit{FInf}^{\succ,S}$ and $\mathit{Red}^{\mathcal{G}^T}$. So, by the previous lemma, $\mathcal{G}^T(N)$ is saturated w.r.t. $\mathit{GInf}^{\succ,T}$ and $\mathit{Red}^T$. Since $\bot \notin \mathcal{G}^T(N)$, the static refutational completeness of $\mathit{GInf}^{\succ,T}$ implies $\mathcal{G}^T(N) \not\models \{\bot\} = \mathcal{G}^T(\{\bot\})$, hence $N \not\models^{\mathcal{G}^T} \{\bot\}$, and hence $N \not\models_{\mathsf{H}} \bot$. □

The quantification over all $T \in \mathrm{gs}(S)$ in the definition of $\mathit{Red}^{\cap}$ looks rather complicated. We observe, however, that $\mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T}$ doesn't depend on $T$ anyhow, so that $\mathit{Red}_{\mathrm{F}}^{\cap}$ agrees with $\mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T}$ for an arbitrary $T$. For $\mathit{Red}_{\mathrm{I}}^{\cap}$, restricting to a single $T \in \mathrm{gs}(S)$ *globally* does not work, but we can at least restrict ourselves to a single $T \in \mathrm{gs}(S)$ *per pre-instance.* We need two simple lemmas:

**Lemma 14.** *Let $\iota$ be an inference in $\mathit{FInf}^{\succ,S}$, let $\iota'$ be a pre-instance $(C_1\theta, C_0\theta)$ or $(C_2\rho\theta, C_1\theta, C_0\theta)$ of $\iota$. Let $T, T' \in \mathrm{gs}(S)$ be two groundings of $S$ that agree on $C_1\theta$ (or $C_2\rho\theta$ and $C_1\theta$, respectively). Then $\iota' \in \mathcal{G}^T(\iota)$ if and only if $\iota' \in \mathcal{G}^{T'}(\iota)$.*

*Proof.* Since $T$ and $T'$ agree on $C_1\theta$ (and $C_2\rho\theta$), we know that $\iota' \in \mathcal{G}^T(\iota)$ if and only if $\iota' \in \mathit{GInf}^{\succ,T}$ and analogously $\iota' \in \mathcal{G}^{T'}(\iota)$ if and only if $\iota' \in \mathit{GInf}^{\succ,T'}$. The side conditions of all ground inferences depend only on the ordering (which is the same for $\mathit{GInf}^{\succ,T}$ and $\mathit{GInf}^{\succ,T'}$) and the selected literals *in the premise(s)* $C_1\theta$ (and $C_2\rho\theta$), which are the same by assumption. So $\iota'$ is an inference in $\mathit{GInf}^{\succ,T}$ if and only if it is an inference in $\mathit{GInf}^{\succ,T'}$. □

**Lemma 15.** *Let $N \subseteq F$, let $\iota$ be an inference in $\mathit{FInf}^{\succ,S}(N)$, let $\iota'$ be a pre-instance of $\iota$. Let $T, T' \in \mathrm{gs}(S)$ be two groundings of $S$ such that $\iota'$ is contained in both $\mathcal{G}^T(\iota)$ and $\mathcal{G}^{T'}(\iota)$. Then $\iota' \in \mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$ if and only if $\iota' \in \mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$.*

*Proof.* First, we observe that the definition of $\mathcal{G}^T(N)$ does not depend on $T$, so that $\mathcal{G}^T(N) = \mathcal{G}^{T'}(N)$. So the set $M$ of all clauses in $\mathcal{G}^T(N)$ that are smaller than the right (or only) premise of $\iota'$ agrees with the set $M'$ of all clauses in $\mathcal{G}^{T'}(N)$ that are smaller than the right (or only) premise of $\iota'$. Consequently $\iota' \in \mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$ if and only if $M \models \mathit{concl}(\iota')$ if and only if $M' \models \mathit{concl}(\iota')$ if and only if $\iota' \in \mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$. □

We can now show that $\mathit{Red}_{\mathrm{I}}^{\cap}$ is in fact equivalent to the classical definition from [3] (in a slightly rephrased form):

**Theorem 16.** *For any inference $\iota \in \mathit{FInf}^{\succ,S}$ of the form $(C_1, C_0)$ or $(C_2, C_1, C_0)$ and for any pre-instance $\iota'$ of $\iota$ of the form $(C_1\theta, C_0\theta)$ or $(C_2\rho\theta, C_1\theta, C_0\theta)$ let $\mathrm{inherit}(S, \iota, \iota')$ be an arbitrary but fixed selection function $T \in \mathrm{gs}(S)$ such that $T(C_1\theta) = (S(C_1))\theta$ (and $T(C_2\rho\theta) = (S(C_2))\rho\theta$), if such a selection function exists. Let $\mathrm{inherit}(S, \iota, \iota')$ be undefined otherwise.*

*Then an inference $\iota = (C_1, C_0) \in \mathit{FInf}^{\succ,S}$ is contained in $\mathit{Red}_{\mathrm{I}}^{\cap}(N)$ if and only if every pre-instance $\iota'$ is contained in $\mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$ whenever $T = \mathrm{inherit}(S, \iota, \iota')$ is defined and $\iota'$ is a $T$-ground instance of $\iota$.*

*Proof.* Let $\iota \in \mathit{FInf}^{\succ,S}$ be contained in $\mathit{Red}_{\mathrm{I}}^{\cap}(N) = \bigcap_{T' \in \mathrm{gs}(S)} \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T'}}(N)$. Then $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T'}}(N)$ for every $T' \in \mathrm{gs}(S)$. Now assume that $T = \mathrm{inherit}(S, \iota, \iota')$ is defined and that $\iota'$ is a $T$-ground instance of $\iota$. Then $T \in \mathrm{gs}(S)$, so $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T}}(N)$, and hence $\iota' \in \mathcal{G}^T(\iota) \subseteq \mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$.

Conversely assume that every pre-instance $\iota'$ is contained in $\mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$ whenever $T = \mathrm{inherit}(S, \iota, \iota')$ is defined and $\iota'$ is a $T$-ground instance of $\iota$. We have to show that $\iota \in \mathit{Red}_{\mathrm{I}}^{\cap}(N)$, which is equivalent to $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T'}}(N)$ for every $T' \in \mathrm{gs}(S)$. Choose $T' \in \mathrm{gs}(S)$ arbitrarily. We now have to show that $\mathcal{G}^{T'}(\iota) \subseteq \mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$, which means by definition that every $T'$-ground instance of $\iota$ is contained in $\mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$. Let $\iota''$ be a $T'$-ground instance of $\iota$, then $T'(C_1\theta) = (S(C_1))\theta$ (and $T'(C_2\rho\theta) = (S(C_2))\rho\theta$), so $T = \mathrm{inherit}(S, \iota, \iota'') \in \mathrm{gs}(S)$ is defined. Since $T(C_1\theta) = T'(C_1\theta) = (S(C_1))\theta$ (and $T(C_2\rho\theta) = T'(C_2\rho\theta) = (S(C_2))\rho\theta$), we know by Lemma 14 that $\iota''$ is a $T$-ground instance of $\iota$. By assumption, $\iota'' \in \mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$, so by Lemma 15, $\iota'' \in \mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$ as required. □

The function $\text{inherit}(S, \iota, \iota')$ is undefined if the selections of the premises of $\iota$ are contradictory for the pre-instance $\iota'$. For instance, consider the clauses $C_2 = (\neg\, f(a) \approx b \vee f(y) \approx y)$ and $C_1 = (\neg\, f(x) \approx b \vee f(x) \approx a)$, where $S$ selects the first literal in $C_1$ and nothing in $C_2$. Then there is a Negative Superposition inference $\iota$

$$\frac{\neg\, f(a) \approx b \vee f(y) \approx y \qquad \neg\, f(x) \approx b \vee f(x) \approx a}{\neg\, f(a) \approx b \vee f(x) \approx a \vee \neg\, x \approx b}$$

in which the maximal second literal of $C_2$ and the selected first literal of $C_1$ are overlapped.

For the pre-instance $\iota'$

$$\frac{\neg\, f(a) \approx b \vee f(a) \approx a \qquad \neg\, f(a) \approx b \vee f(a) \approx a}{\neg\, f(a) \approx b \vee f(a) \approx a \vee \neg\, a \approx b}$$

of $\iota$ there is no selection function $T$ such that $T(C_1\theta) = (S(C_1))\theta$ and $T(C_2\rho\theta) = (S(C_2))\rho\theta$, since $C_1\theta = C_2\rho\theta$, but $(S(C_1))\theta \neq (S(C_2))\rho\theta$. (In fact, $\iota'$ violates the ordering restrictions of Negative Superposition, so it is not an inference for any selection function.)

To see that restricting to a single $T \in \text{gs}(S)$ *globally* does not work, consider the clauses $C_2 = (\neg\, f(x) \approx g(a, x') \vee f(f(x)) \approx g(a, x'))$ and $C_1 = (\neg\, f(y) \approx g(y', a) \vee f(f(y)) \approx g(y', a))$, where $S$ selects the first literal in $C_1$ and nothing in $C_2$ and the term ordering $\succ$ is an LPO with precedence $f > g > a$. There is a Negative Superposition inference $\iota$

$$\frac{\neg\, f(x) \approx g(a, x') \vee f(f(x)) \approx g(a, x') \qquad \neg\, f(y) \approx g(y', a) \vee f(f(y)) \approx g(y', a)}{\neg\, f(x) \approx g(a, x') \vee \neg\, g(a, x') \approx g(y', a) \vee f(f(f(x))) \approx g(y', a)}$$

in which the maximal second literal of $C_2$ and the selected first literal of $C_1$ are overlapped.

Now consider the pre-instances $\iota_1$

$$\frac{\begin{array}{l}\neg\, f(a) \approx g(a, a) \vee f(f(a)) \approx g(a, a) \\ \qquad\qquad\qquad\qquad \neg\, f(f(a)) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a)\end{array}}{\neg\, f(a) \approx g(a, a) \vee \neg\, g(a, a) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a)}$$

and $\iota_2$

$$\frac{\begin{array}{l}\neg\, f(f(a)) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a) \\ \qquad\qquad\qquad\qquad \neg\, f(f(f(a))) \approx g(a, a) \vee f(f(f(f(a)))) \approx g(a, a)\end{array}}{\neg\, f(f(a)) \approx g(a, a) \vee \neg\, g(a, a) \approx g(a, a) \vee f(f(f(f(a)))) \approx g(a, a)}$$

of $\iota$. The ground clause $D = (\neg\, f(f(a)) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a))$ occurs in $\iota_1$ as a ground instance of $C_1$ and in $\iota_2$ as a ground instance of $C_2$. Since the selection in the ground instances of $C_1$ and $C_2$ must correspond to the selection in $C_1$ and $C_2$ themselves, this means that for $\iota_1$ the first literal in $D$ must be selected and for $\iota_2$ no literal in $D$ must be selected. Consequently, the selection functions for $\iota_1$ and $\iota_2$ must be different. (In fact, $\iota_i$ is a $T_i$-instances of $\iota$ with $T_i = \text{inherit}(S, \iota, \iota_i)$ for both $i = 1, 2$.)

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